Lecture 10

Dr.
Mahmoud Khedr

Third Edition

CHAPTER

7

MECHANICS OF MATERIALS

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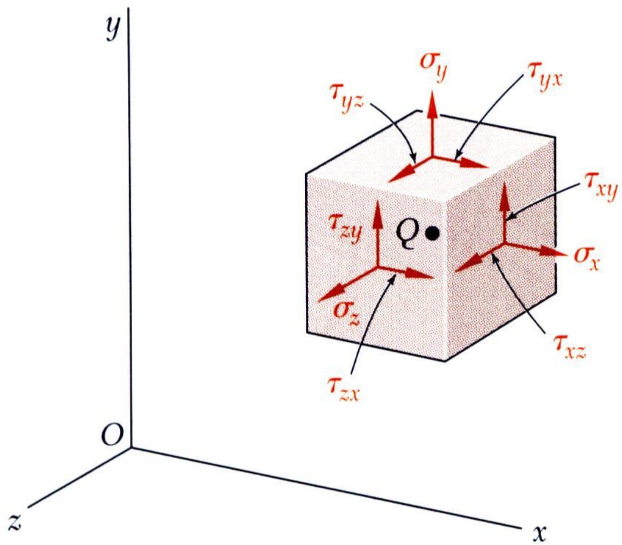
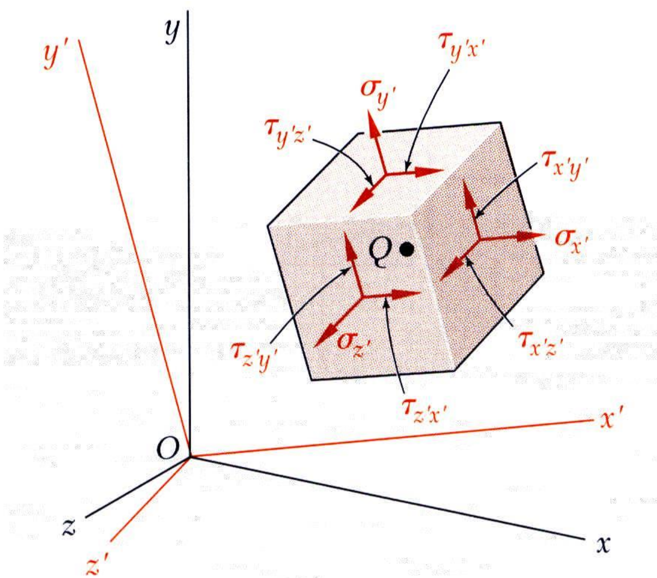
Transformations of Stress

Introduction





- The most general state of stress at a point may be represented by 6 components,

  $\sigma_x, \sigma_y, \sigma_z$ normal stresses

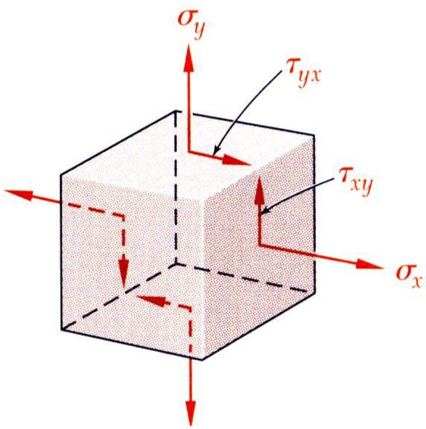
  $\tau_{xy},\ \tau_{yz},\ \tau_{zx}$ shearing stresses

  (Note: $\tau_{xy} = \tau_{yx},\ \tau_{yz} = \tau_{zy},\ \tau_{zx} = \tau_{xz}$)

- Same state of stress is represented by a different set of components if axes are rotated.

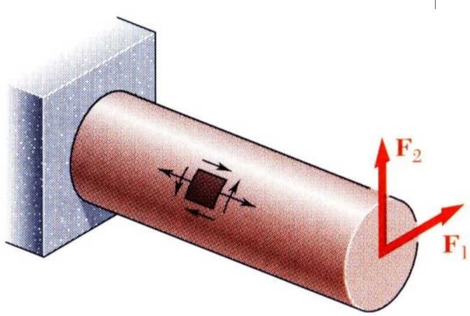
- The first part of the chapter is concerned with how the components of stress are transformed under a rotation of the coordinate axes. The second part of the chapter is devoted to a similar analysis of the transformation of the components of strain.

Introduction



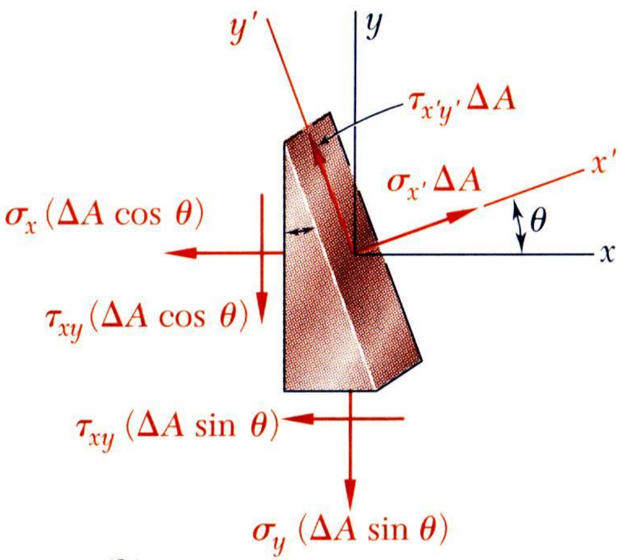
- *Plane Stress* - state of stress in which two faces of the cubic element are free of stress. For the illustrated example, the state of stress is defined by
  $\sigma_x, \sigma_y, \tau_{xy}$ and $\sigma_z = \tau_{zx} = \tau_{zy} = 0.$

- State of plane stress occurs in a thin plate subjected to forces acting in the midplane of the plate.



- State of plane stress also occurs on the free surface of a structural element or machine component, i.e., at any point of the surface not subjected to an external force.

Transformation of Plane Stress

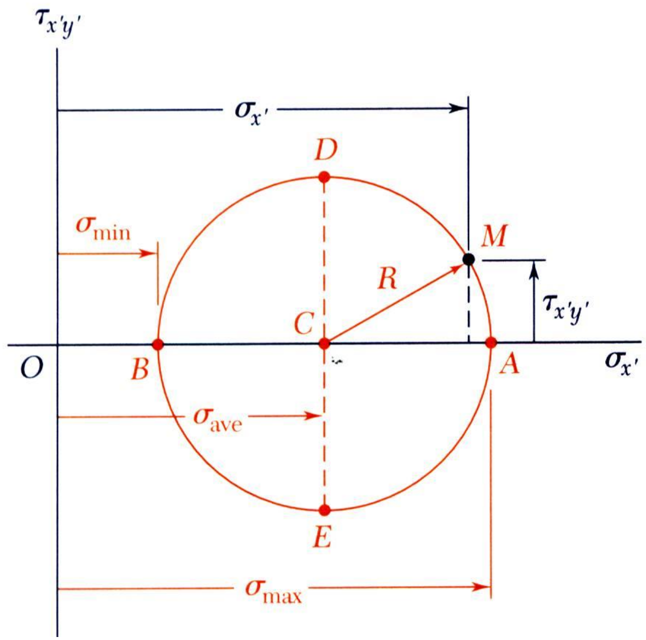


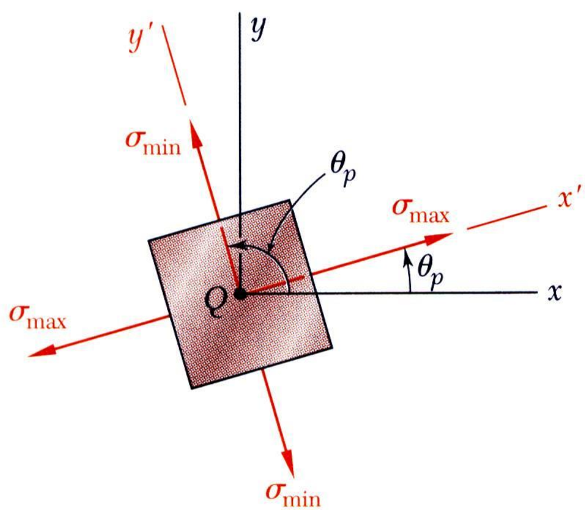
- Consider the conditions for equilibrium of a prismatic element with faces perpendicular to the *x*, *y*, and *x'* axes.

$$\sum F_{x'} = 0 = \sigma_{x'}\Delta A - \sigma_x(\Delta A\cos\theta)\cos\theta - \tau_{xy}(\Delta A\cos\theta)\sin\theta$$
$$-\sigma_y(\Delta A\sin\theta)\sin\theta - \tau_{xy}(\Delta A\sin\theta)\cos\theta$$

$$\sum F_{y'} = 0 = \tau_{x'y'}\Delta A + \sigma_x(\Delta A\cos\theta)\sin\theta - \tau_{xy}(\Delta A\cos\theta)\cos\theta$$
$$-\sigma_y(\Delta A\sin\theta)\cos\theta + \tau_{xy}(\Delta A\sin\theta)\sin\theta$$

Principal Stresses





- The previous equations are combined to yield parametric equations for a circle,

$$\left(\sigma_{x'} - \sigma_{ave}\right)^2 + \tau_{x'y'}^2 = R^2$$

where

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

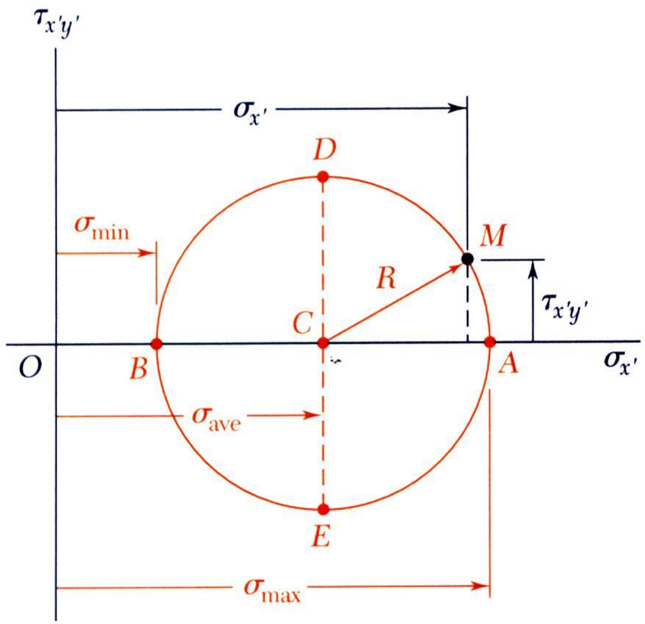
- *Principal stresses* occur on the *principal planes of stress* with zero shearing stresses.

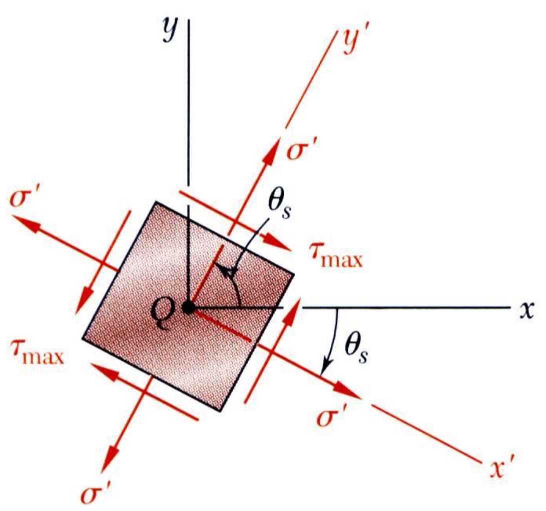
$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Note : defines two angles separated by 90^{o}

Maximum Shearing Stress





Maximum shearing stress occurs for $\sigma_{x'} = \sigma_{ave}$

$$\tau_{\max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

Note : defines two angles separated by 90° and offset from θ_p by 45°

$$\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

Example 7.01

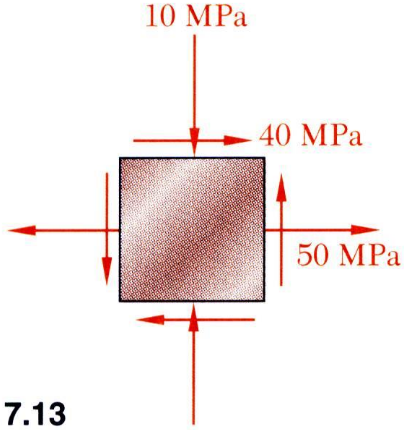


Fig. 7.13

For the state of plane stress shown, determine (a) the principal panes, (b) the principal stresses, (c) the maximum shearing stress and the corresponding normal stress.

SOLUTION:

- Find the element orientation for the principal stresses from

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

- Determine the principal stresses from

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- Calculate the maximum shearing stress with

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma' = \frac{\sigma_x + \sigma_y}{2}$$

Example 7.01

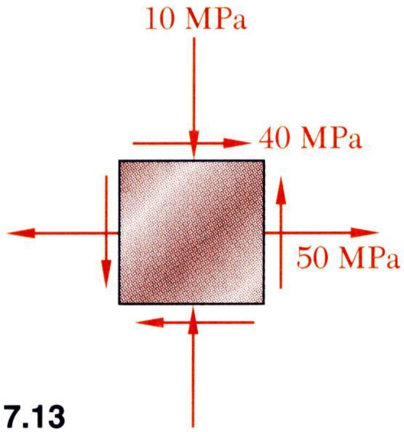


Fig. 7.13

$$\sigma_x = +50\,\text{MPa} \qquad \tau_{xy} = +40\,\text{MPa}$$

$$\sigma_x = -10\,\text{MPa}$$

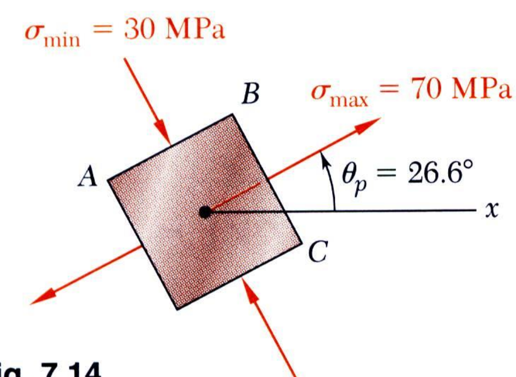


Fig. 7.14

SOLUTION:

- Find the element orientation for the principal stresses from

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(+40)}{50-(-10)} = 1.333$$

$$2\theta_p = 53.1°,\ 233.1°$$

$$\boxed{\theta_p = 26.6°,\ 116.6°}$$

- Determine the principal stresses from

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 20 \pm \sqrt{(30)^2 + (40)^2}$$

$$\boxed{\begin{aligned}\sigma_{\max} &= 70\,\text{MPa}\\ \sigma_{\min} &= -30\,\text{MPa}\end{aligned}}$$

Example 7.01

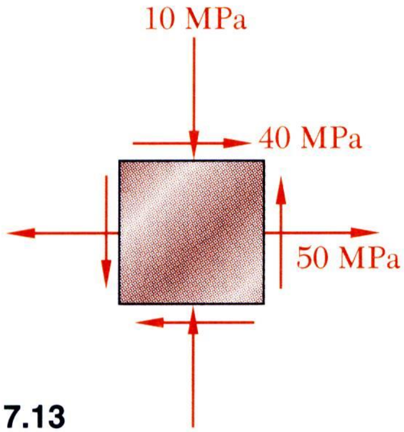


Fig. 7.13

$$\sigma_x = +50\,\text{MPa} \qquad \tau_{xy} = +40\,\text{MPa}$$

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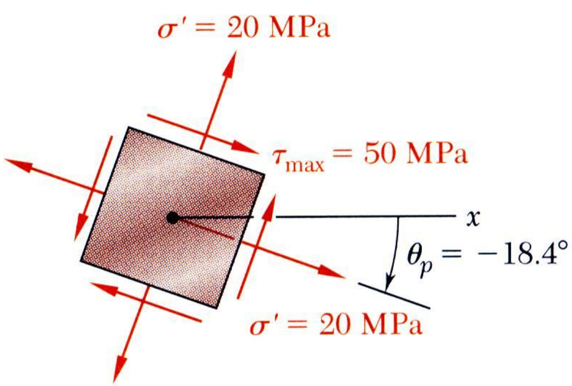


Fig. 7.16

- Calculate the maximum shearing stress with

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{(30)^2 + (40)^2}$$

$$\boxed{\tau_{max} = 50\,\text{MPa}}$$

$$\theta_s = \theta_p - 45$$

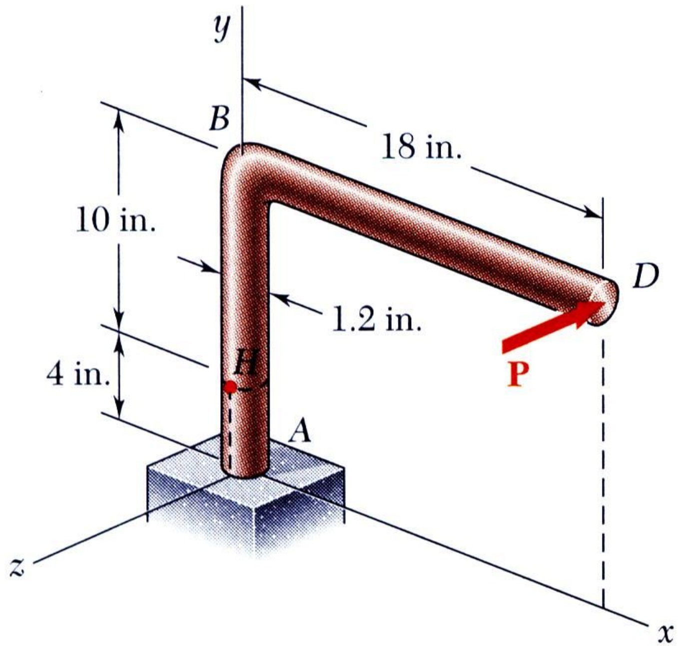
$$\boxed{\theta_s = -18.4°,\ 71.6°}$$

- The corresponding normal stress is

$$\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{50 - 10}{2}$$

$$\boxed{\sigma' = 20\,\text{MPa}}$$

Sample Problem 7.1

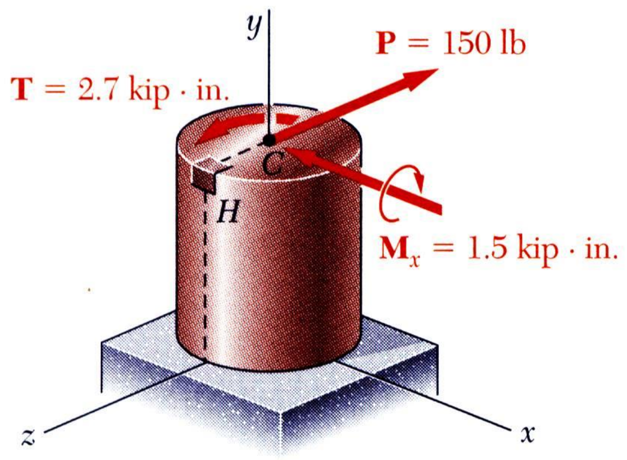


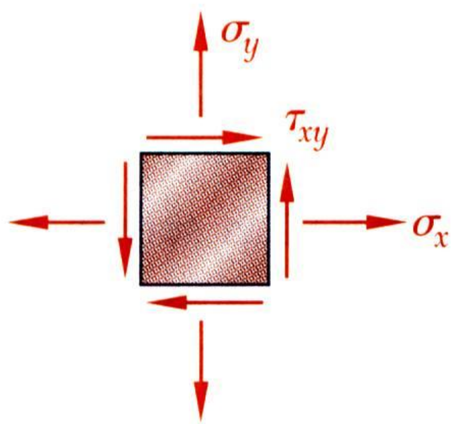
A single horizontal force *P* of 150 lb magnitude is applied to end D of lever *ABD*. Determine (a) the normal and shearing stresses on an element at point *H* having sides parallel to the *x* and *y* axes, (b) the principal planes and principal stresses at the point *H*.

SOLUTION:

- Determine an equivalent force-couple system at the center of the transverse section passing through *H*.
- Evaluate the normal and shearing stresses at *H*.
- Determine the principal planes and calculate the principal stresses.

Sample Problem 7.1





SOLUTION:

- Determine an equivalent force-couple system at the center of the transverse section passing through *H*.

$$P = 150\,\text{lb}$$

$$T = (150\,\text{lb})(18\,\text{in}) = 2.7\,\text{kip}\cdot\text{in}$$

$$M_x = (150\,\text{lb})(10\,\text{in}) = 1.5\,\text{kip}\cdot\text{in}$$

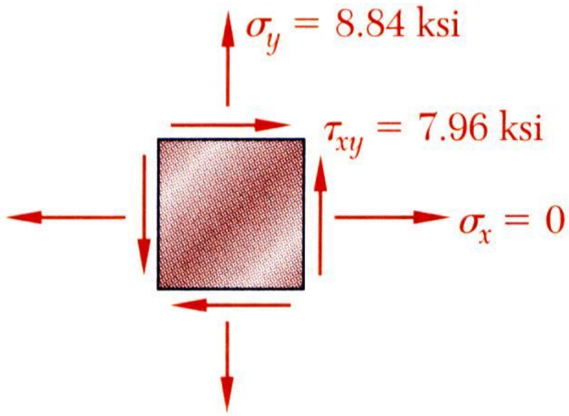
- Evaluate the normal and shearing stresses at *H*.

$$\sigma_y = +\frac{Mc}{I} = +\frac{(1.5\,\text{kip}\cdot\text{in})(0.6\,\text{in})}{\frac{1}{4}\pi(0.6\,\text{in})^4}$$

$$\tau_{xy} = +\frac{Tc}{J} = +\frac{(2.7\,\text{kip}\cdot\text{in})(0.6\,\text{in})}{\frac{1}{2}\pi(0.6\,\text{in})^4}$$

$$\boxed{\sigma_x = 0 \quad \sigma_y = +8.84\,\text{ksi} \quad \tau_y = +7.96\,\text{ksi}}$$

Sample Problem 7.1



- Determine the principal planes and calculate the principal stresses.

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(7.96)}{0-8.84} = -1.8$$

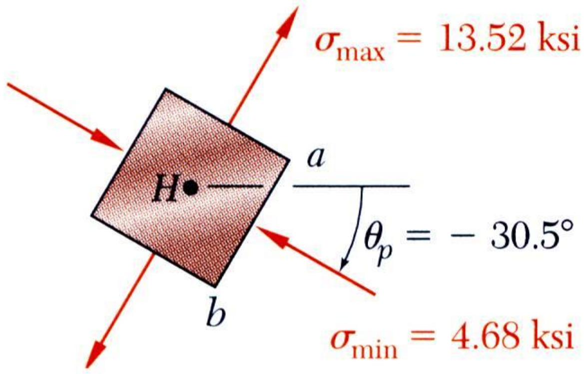
$$2\theta_p = -61.0°, 119°$$

$$\boxed{\theta_p = -30.5°,\ 59.5°}$$

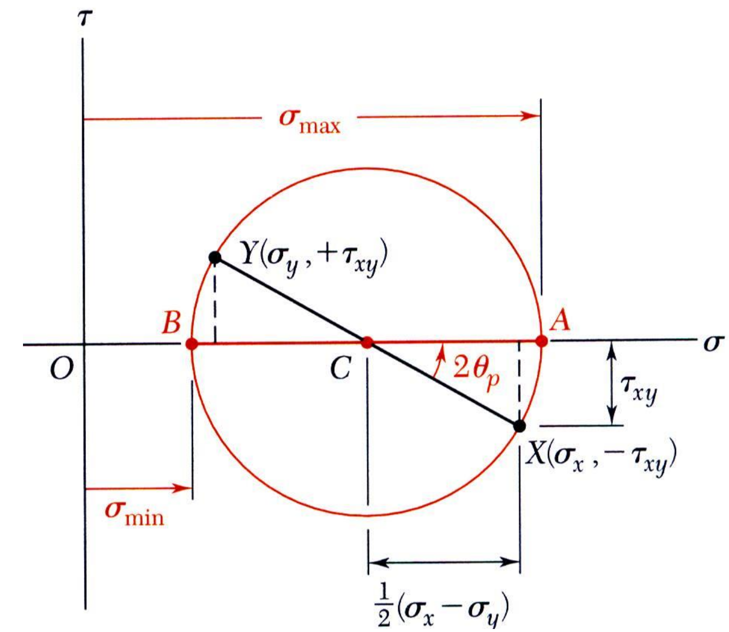
$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

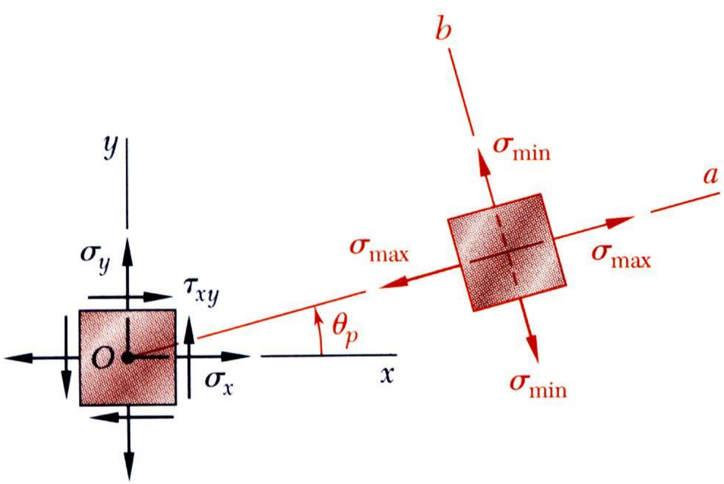
$$= \frac{0+8.84}{2} \pm \sqrt{\left(\frac{0-8.84}{2}\right)^2 + (7.96)^2}$$

$$\boxed{\begin{aligned}\sigma_{\max} &= +13.52\,\text{ksi}\\ \sigma_{\min} &= -4.68\,\text{ksi}\end{aligned}}$$



Mohr's Circle for Plane Stress





- With the physical significance of Mohr's circle for plane stress established, it may be applied with simple geometric considerations. Critical values are estimated graphically or calculated.

- For a known state of plane stress $\sigma_x, \sigma_y, \tau_{xy}$ plot the points X and Y and construct the circle centered at C.

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

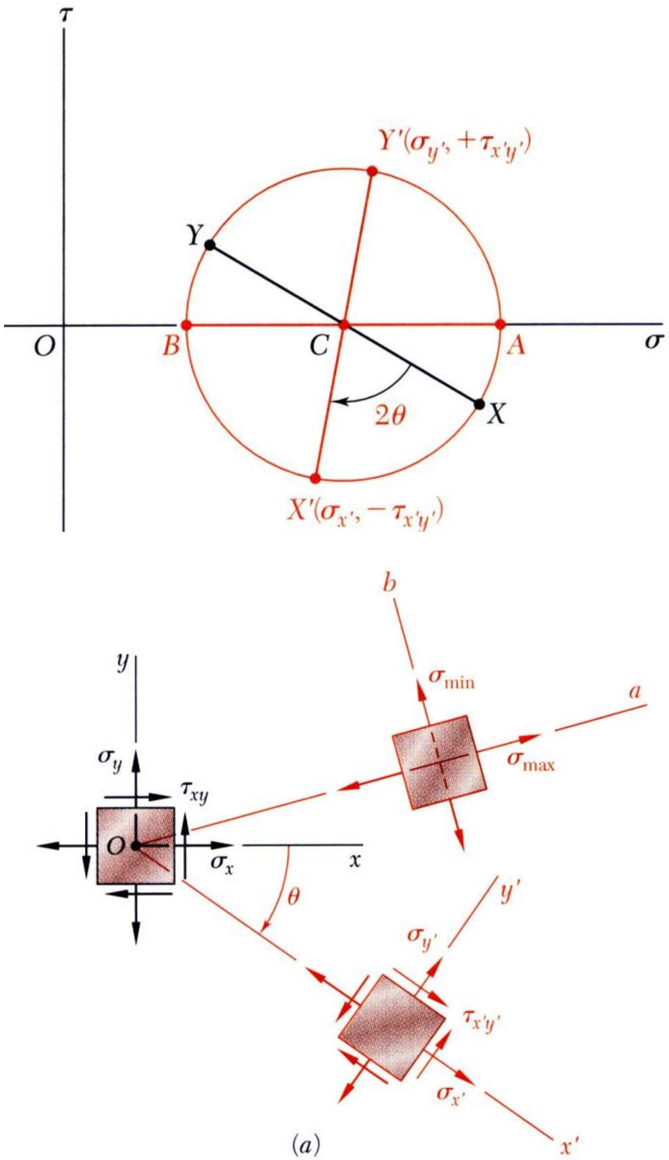
- The principal stresses are obtained at A and B.

$$\sigma_{\max,\min} = \sigma_{ave} \pm R$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

The direction of rotation of Ox to Oa is the same as CX to CA.

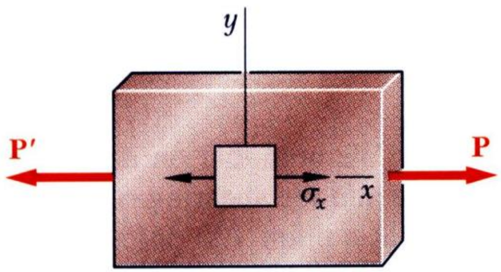
Mohr’s Circle for Plane Stress

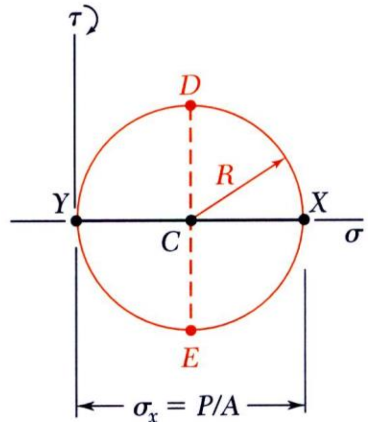


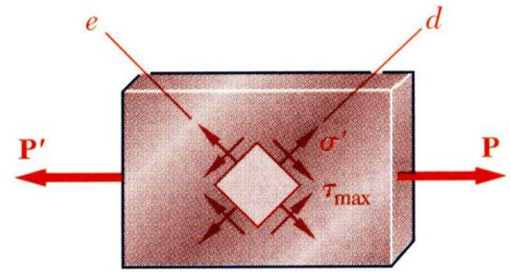
- With Mohr’s circle uniquely defined, the state of stress at other axes orientations may be depicted.
- For the state of stress at an angle θ with respect to the *xy* axes, construct a new diameter *X’Y’* at an angle 2θ with respect to *XY*.
- Normal and shear stresses are obtained from the coordinates *X’Y’*.

Mohr’s Circle for Plane Stress

- Mohr’s circle for centric axial loading:



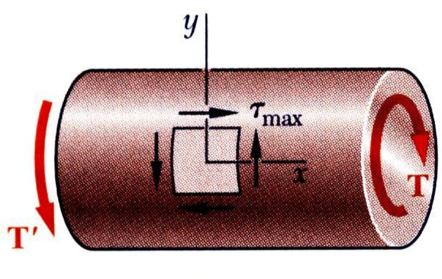


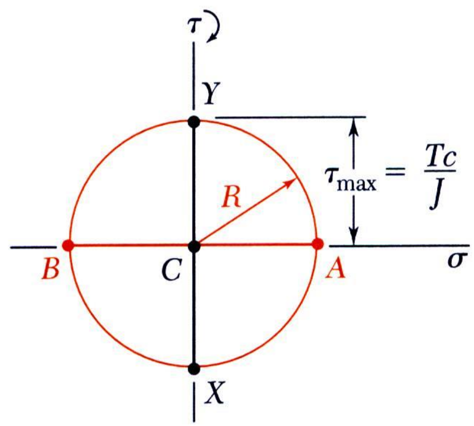


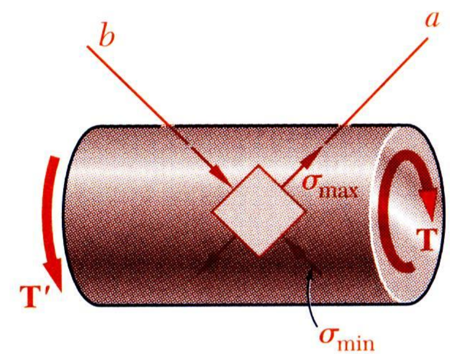
$$\sigma_x = \frac{P}{A}, \quad \sigma_y = \tau_{xy} = 0$$

$$\sigma_x = \sigma_y = \tau_{xy} = \frac{P}{2A}$$

- Mohr’s circle for torsional loading:







$$\sigma_x = \sigma_y = 0 \quad \tau_{xy} = \frac{Tc}{J}$$

$$\sigma_x = \sigma_y = \frac{Tc}{J} \quad \tau_{xy} = 0$$

Example 7.02

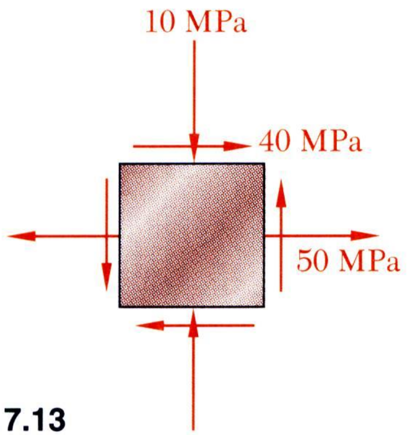
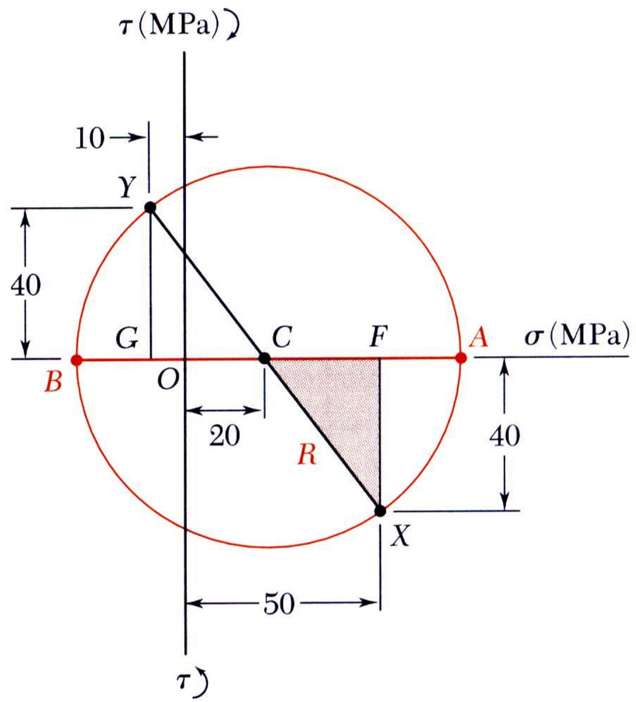


Fig. 7.13



For the state of plane stress shown, (a) construct Mohr's circle, determine (b) the principal planes, (c) the principal stresses, (d) the maximum shearing stress and the corresponding normal stress.

SOLUTION:

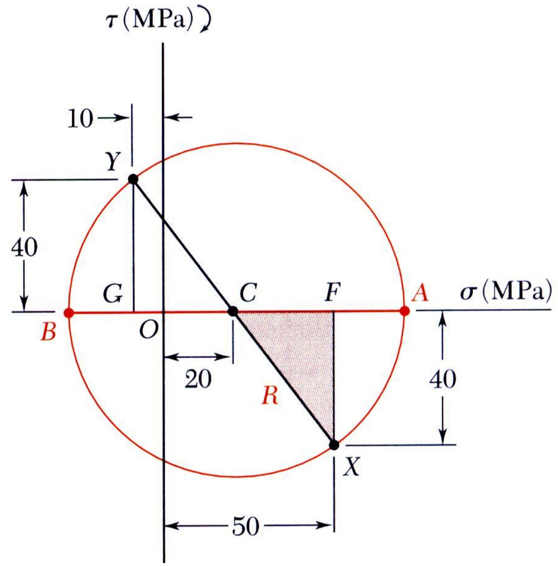
- Construction of Mohr's circle

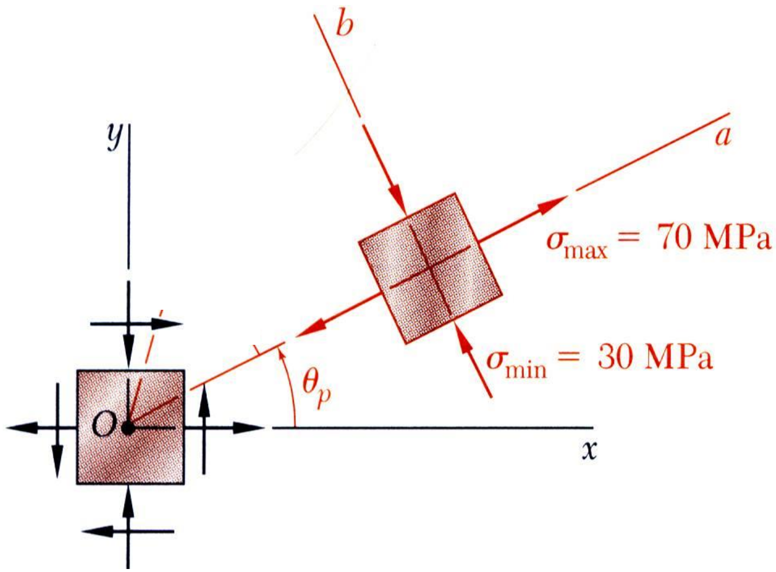
$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{(50)+(-10)}{2} = 20\,\text{MPa}$$

$$CF = 50 - 20 = 30\,\text{MPa} \qquad FX = 40\,\text{MPa}$$

$$R = CX = \sqrt{(30)^2 + (40)^2} = 50\,\text{MPa}$$

Example 7.02





- Principal planes and stresses

$$\sigma_{\max} = OA = OC + CA = 20 + 50$$

$$\boxed{\sigma_{\max} = 70\,\text{MPa}}$$

$$\sigma_{\max} = OB = OC - BC = 20 - 50$$

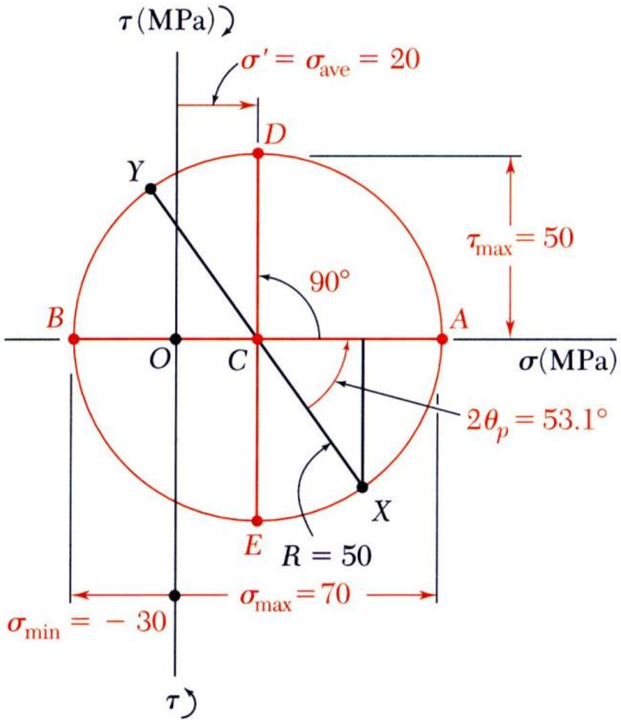
$$\boxed{\sigma_{\max} = -30\,\text{MPa}}$$

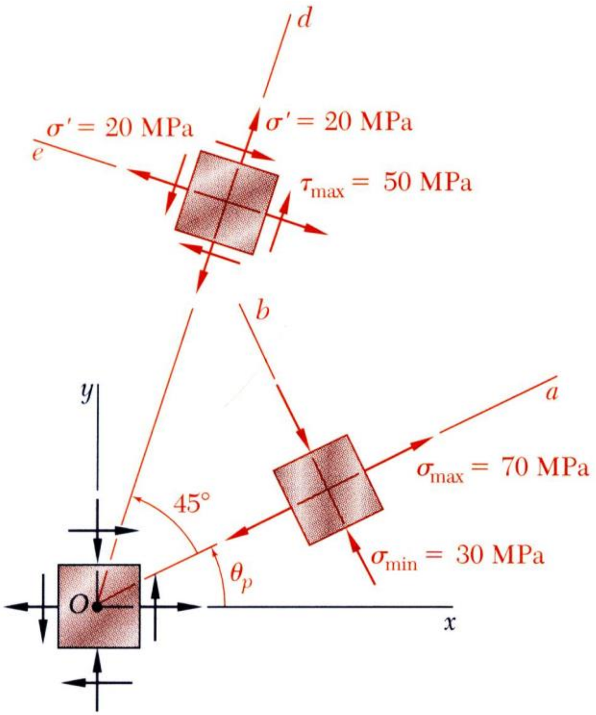
$$\tan 2\theta_p = \frac{FX}{CP} = \frac{40}{30}$$

$$2\theta_p = 53.1°$$

$$\boxed{\theta_p = 26.6°}$$

Example 7.02



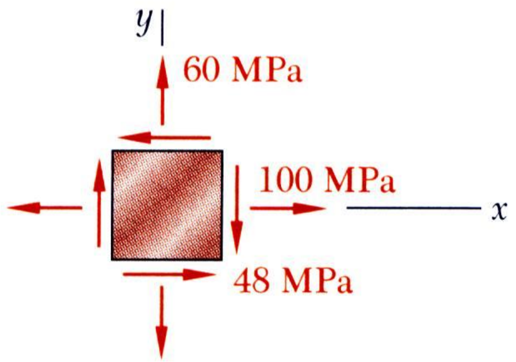


- Maximum shear stress

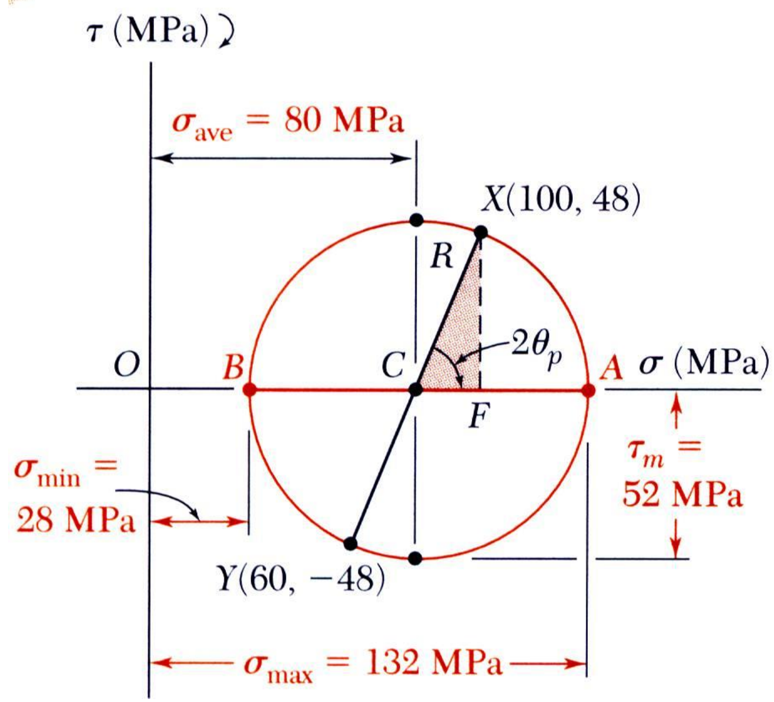
$$\theta_s = \theta_p + 45^\circ \qquad \tau_{\max} = R \qquad \sigma' = \sigma_{ave}$$

$$\boxed{\theta_s = 71.6^\circ} \qquad \boxed{\tau_{\max} = 50\text{ MPa}} \qquad \boxed{\sigma' = 20\text{ MPa}}$$

Sample Problem 7.2



For the state of stress shown, determine (a) the principal planes and the principal stresses, (b) the stress components exerted on the element obtained by rotating the given element counterclockwise through 30 degrees.



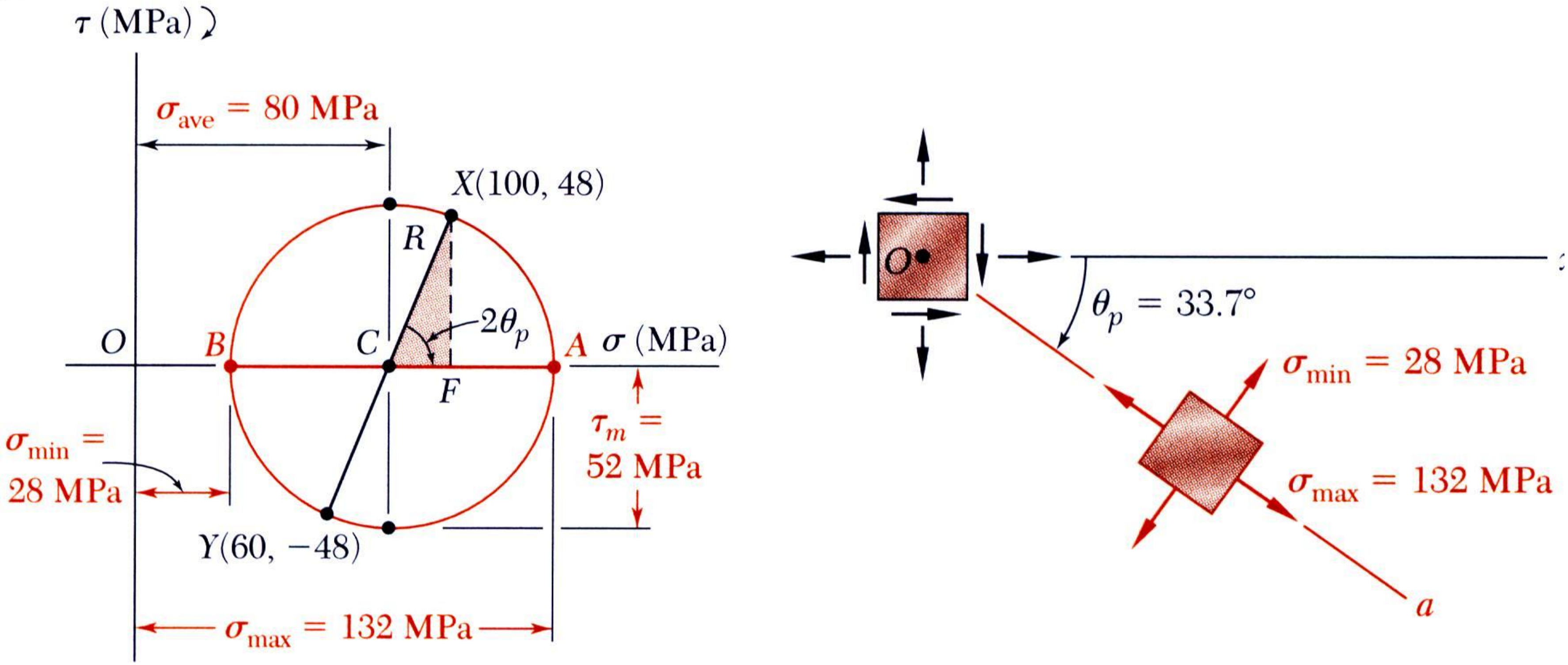
SOLUTION:

- Construct Mohr's circle

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{100 + 60}{2} = 80\,\text{MPa}$$

$$R = \sqrt{(CF)^2 + (FX)^2} = \sqrt{(20)^2 + (48)^2} = 52\,\text{MPa}$$

Sample Problem 7.2



- Principal planes and stresses

$$\tan 2\theta_p = \frac{XF}{CF} = \frac{48}{20} = 2.4$$

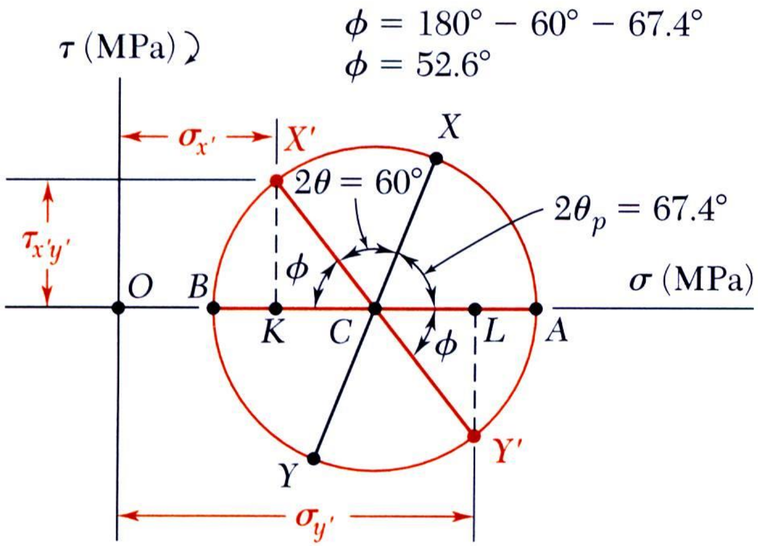
$$2\theta_p = 67.4°$$

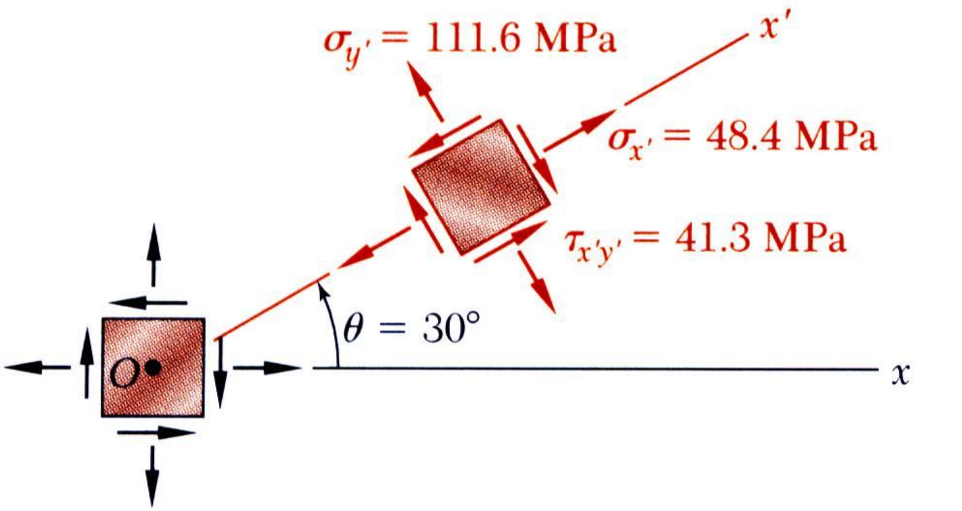
$$\boxed{\theta_p = 33.7° \text{ clockwise}}$$

$$\sigma_{max} = OA = OC + CA$$
$$= 80 + 52$$
$$\boxed{\sigma_{max} = +132\,\text{MPa}}$$

$$\sigma_{max} = OA = OC - BC$$
$$= 80 - 52$$
$$\boxed{\sigma_{min} = +28\,\text{MPa}}$$

Sample Problem 7.2





- Stress components after rotation by 30°

  Points *X'* and *Y'* on Mohr's circle that correspond to stress components on the rotated element are obtained by rotating *XY* counterclockwise through $2\theta = 60°$

$$\phi = 180° - 60° - 67.4° = 52.6°$$

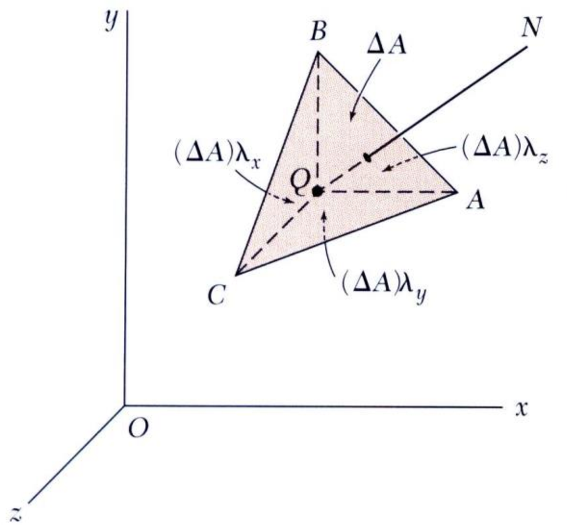
$$\sigma_{x'} = OK = OC - KC = 80 - 52\cos 52.6°$$

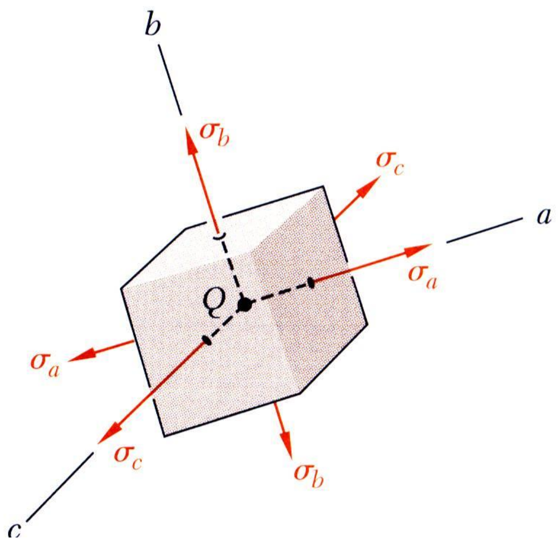
$$\sigma_{y'} = OL = OC + CL = 80 + 52\cos 52.6°$$

$$\tau_{x'y'} = KX' = 52\sin 52.6°$$

$$\boxed{\begin{aligned} \sigma_{x'} &= +48.4\,\text{MPa} \\ \sigma_{y'} &= +111.6\,\text{MPa} \\ \tau_{x'y'} &= 41.3\,\text{MPa} \end{aligned}}$$

General State of Stress





- Consider the general 3D state of stress at a point and the transformation of stress from element rotation
- State of stress at Q defined by: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- Consider tetrahedron with face perpendicular to the line QN with direction cosines: $\lambda_x, \lambda_y, \lambda_z$
- The requirement $\sum F_n = 0$ leads to,

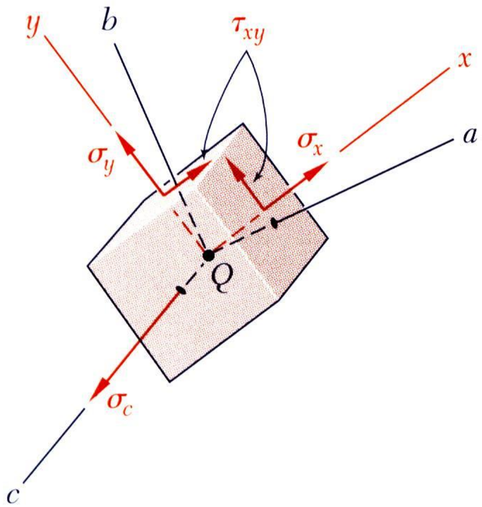
$$\sigma_n = \sigma_x \lambda_x^2 + \sigma_y \lambda_y^2 + \sigma_z \lambda_z^2 + 2\tau_{xy}\lambda_x\lambda_y + 2\tau_{yz}\lambda_y\lambda_z + 2\tau_{zx}\lambda_z\lambda_x$$

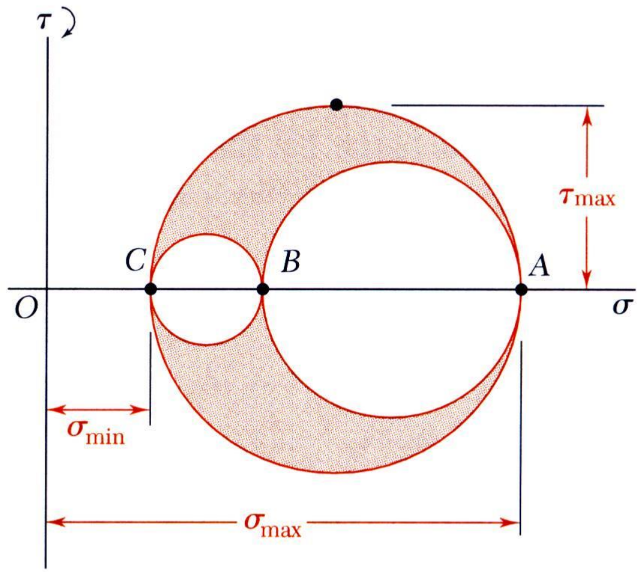
- Form of equation guarantees that an element orientation can be found such that

$$\sigma_n = \sigma_a \lambda_a^2 + \sigma_b \lambda_b^2 + \sigma_c \lambda_c^2$$

These are the principal axes and principal planes and the normal stresses are the principal stresses.

Application of Mohr's Circle to the Three-Dimensional Analysis of Stress

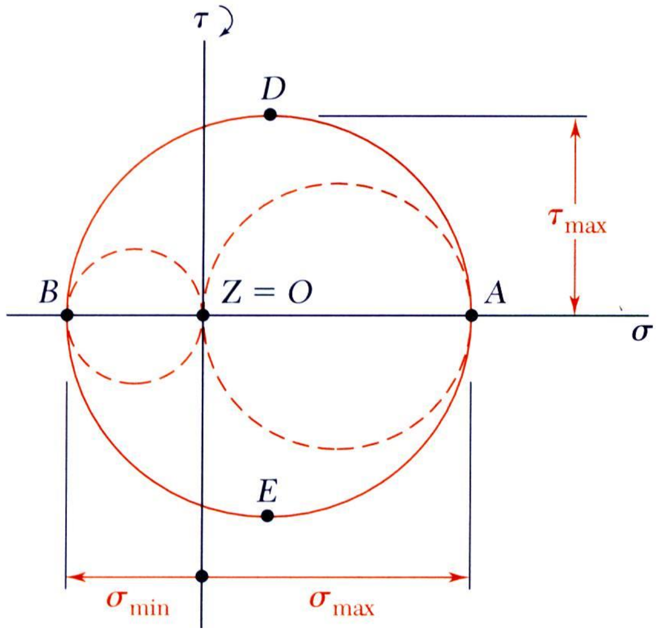


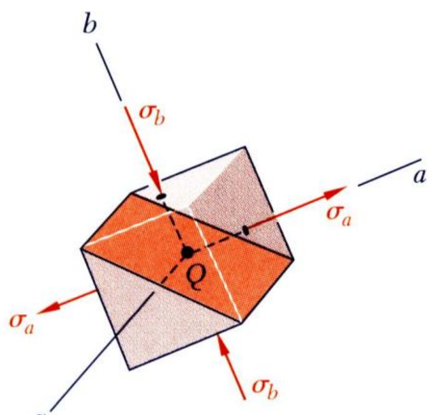


- Transformation of stress for an element rotated around a principal axis may be represented by Mohr's circle.
- Points *A*, *B*, and *C* represent the principal stresses on the principal planes (shearing stress is zero)
- The three circles represent the normal and shearing stresses for rotation around each principal axis.
- Radius of the largest circle yields the maximum shearing stress.

$$\tau_{\max} = \frac{1}{2}\left|\sigma_{\max} - \sigma_{\min}\right|$$

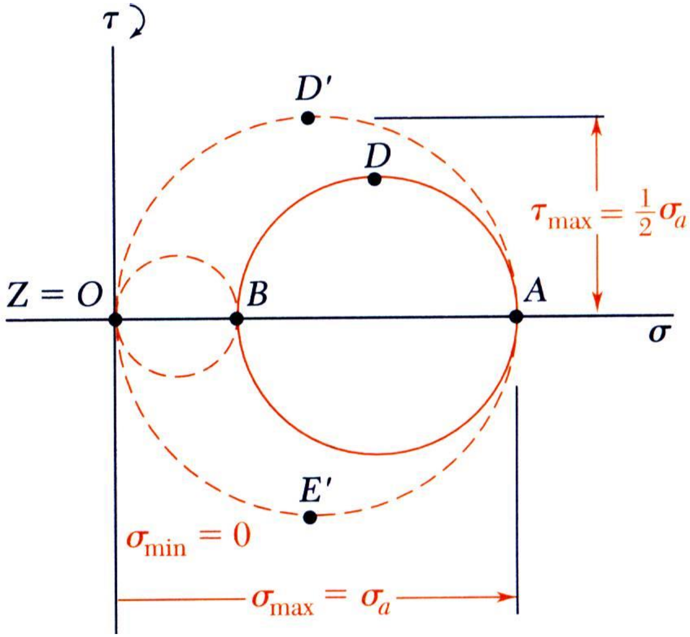
Application of Mohr's Circle to the Three-Dimensional Analysis of Stress

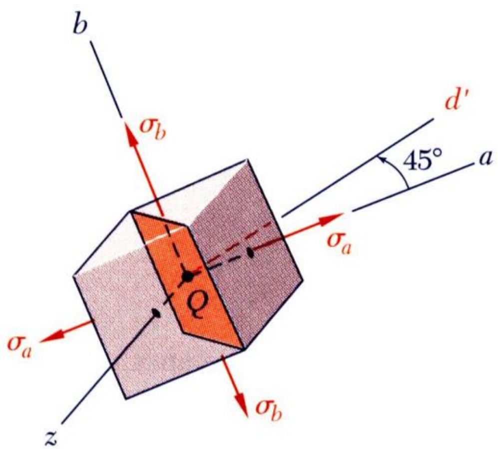




- In the case of plane stress, the axis perpendicular to the plane of stress is a principal axis (shearing stress equal zero).
- If the points *A* and *B* (representing the principal planes) are on opposite sides of the origin, then
    a) the corresponding principal stresses are the maximum and minimum normal stresses for the element
    b) the maximum shearing stress for the element is equal to the maximum "in-plane" shearing stress
    c) planes of maximum shearing stress are at 45° to the principal planes.

Application of Mohr's Circle to the Three-Dimensional Analysis of Stress





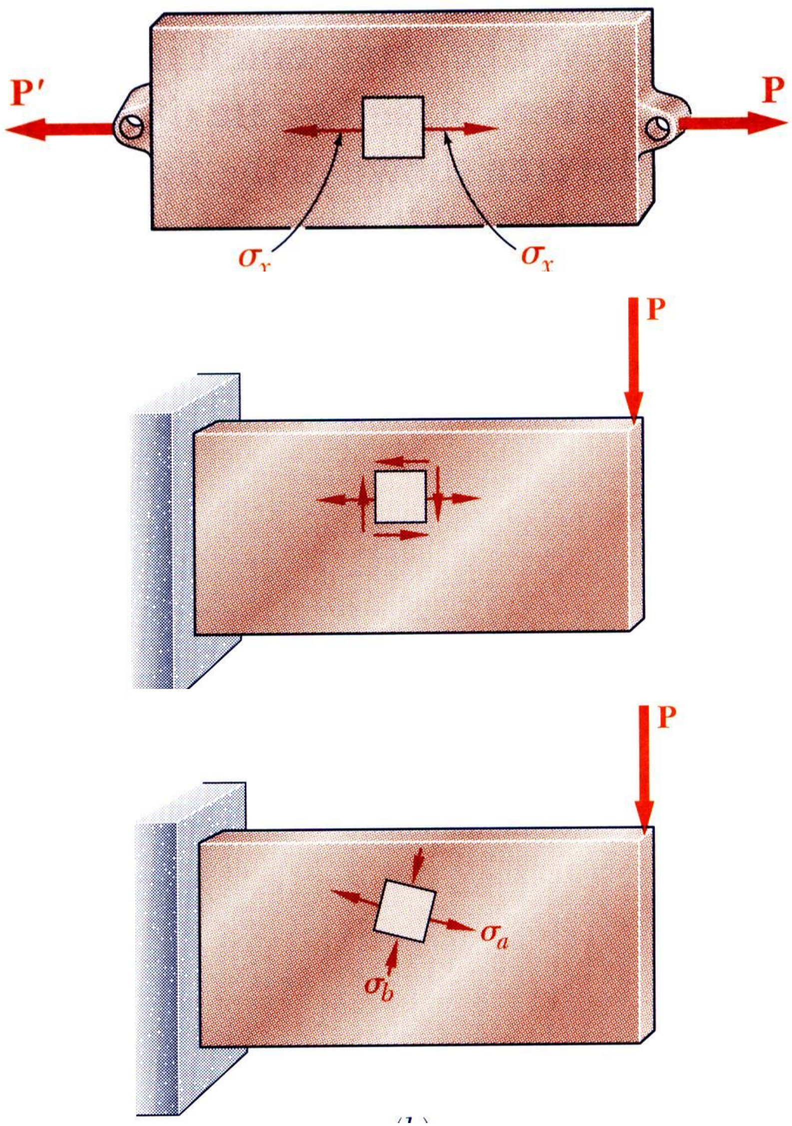
- If A and B are on the same side of the origin (i.e., have the same sign), then

  a) the circle defining σ_{max}, σ_{min}, and τ_{max} for the element is not the circle corresponding to transformations within the plane of stress

  b) maximum shearing stress for the element is equal to half of the maximum stress

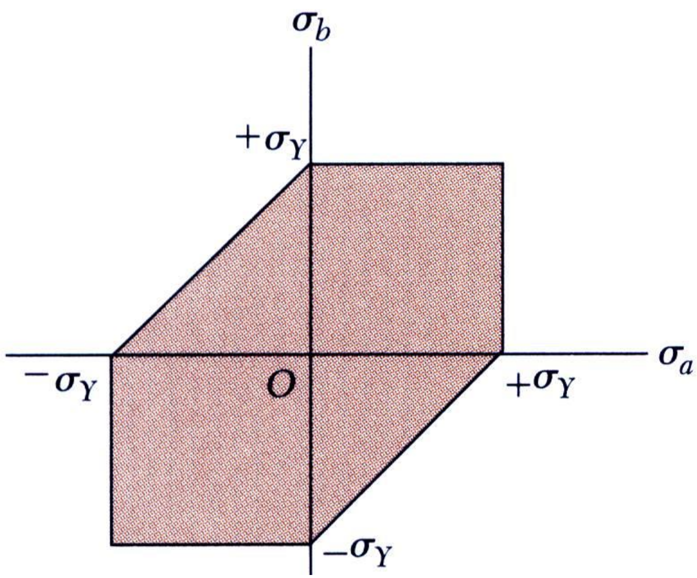
  c) planes of maximum shearing stress are at 45 degrees to the plane of stress

Yield Criteria for Ductile Materials Under Plane Stress



- Failure of a machine component subjected to uniaxial stress is directly predicted from an equivalent tensile test
- Failure of a machine component subjected to plane stress cannot be directly predicted from the uniaxial state of stress in a tensile test specimen
- It is convenient to determine the principal stresses and to base the failure criteria on the corresponding biaxial stress state
- Failure criteria are based on the mechanism of failure. Allows comparison of the failure conditions for a uniaxial stress test and biaxial component loading

Yield Criteria for Ductile Materials Under Plane Stress



Maximum shearing stress criteria:

Structural component is safe as long as the maximum shearing stress is less than the maximum shearing stress in a tensile test specimen at yield, i.e.,

$$\tau_{\max} < \tau_Y = \frac{\sigma_Y}{2}$$

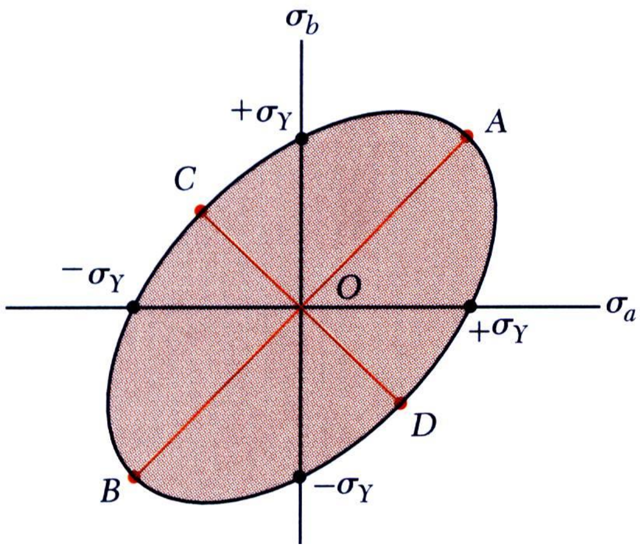
For σ_a and σ_b with the same sign,

$$\tau_{\max} = \frac{|\sigma_a|}{2} \text{ or } \frac{|\sigma_b|}{2} < \frac{\sigma_Y}{2}$$

For σ_a and σ_b with opposite signs,

$$\tau_{\max} = \frac{|\sigma_a - \sigma_b|}{2} < \frac{\sigma_Y}{2}$$

Yield Criteria for Ductile Materials Under Plane Stress



Maximum distortion energy criteria:

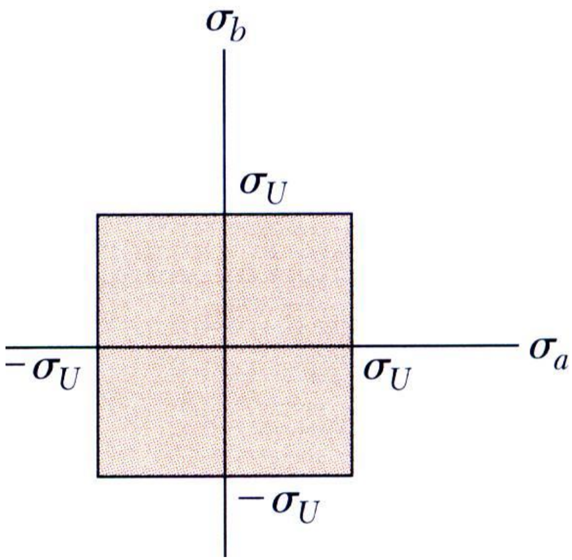
Structural component is safe as long as the distortion energy per unit volume is less than that occurring in a tensile test specimen at yield.

$$u_d < u_Y$$

$$\frac{1}{6G}\left(\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2\right) < \frac{1}{6G}\left(\sigma_Y^2 - \sigma_Y \times 0 + 0^2\right)$$

$$\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 < \sigma_Y^2$$

Fracture Criteria for Brittle Materials Under Plane Stress



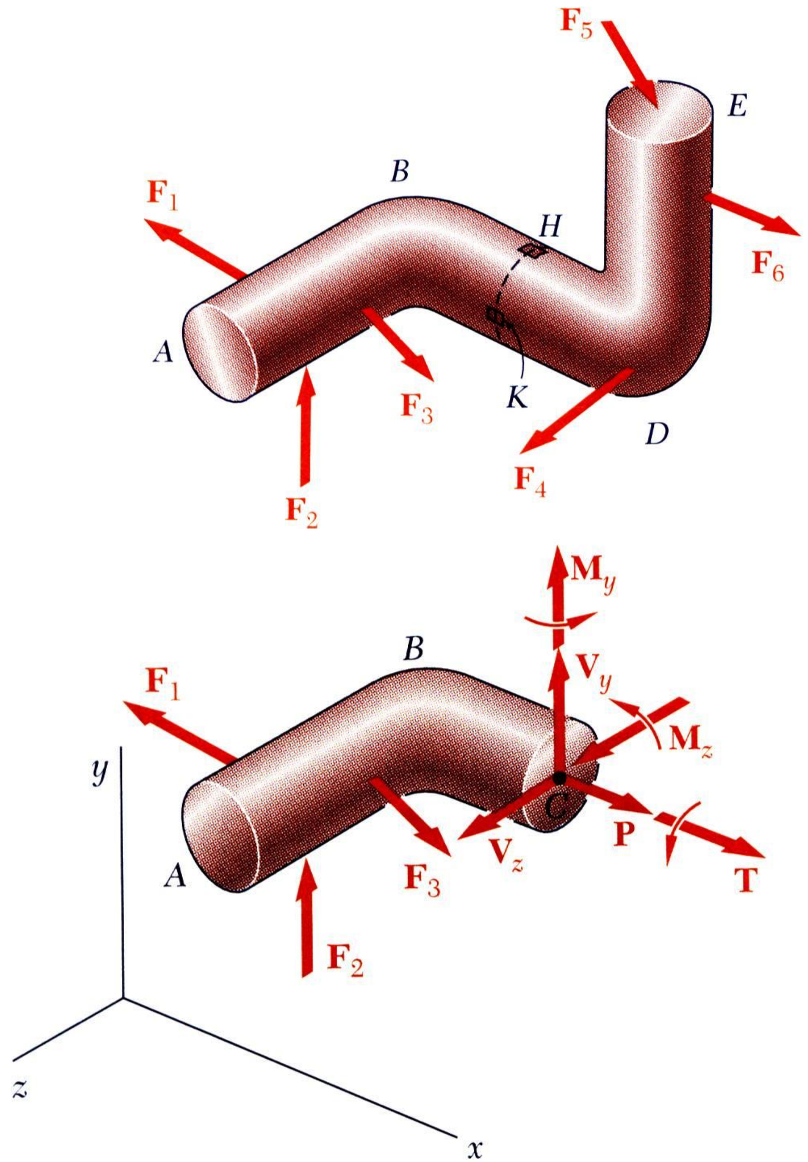
Brittle materials fail suddenly through rupture or fracture in a tensile test. The failure condition is characterized by the ultimate strength σ_U.

Maximum normal stress criteria:

Structural component is safe as long as the maximum normal stress is less than the ultimate strength of a tensile test specimen.

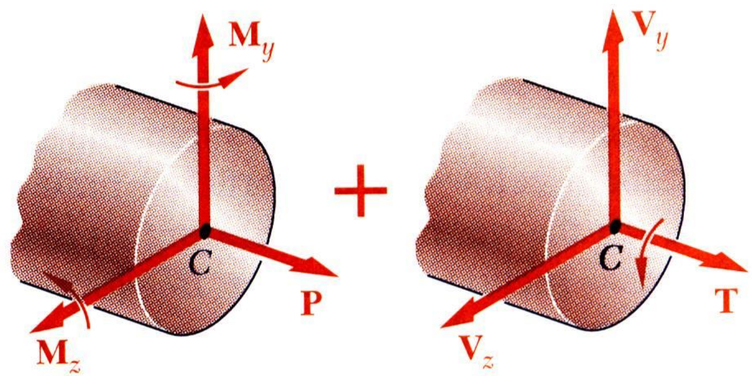
$$|\sigma_a| < \sigma_U$$
$$|\sigma_b| < \sigma_U$$

Stresses Under Combined Loadings



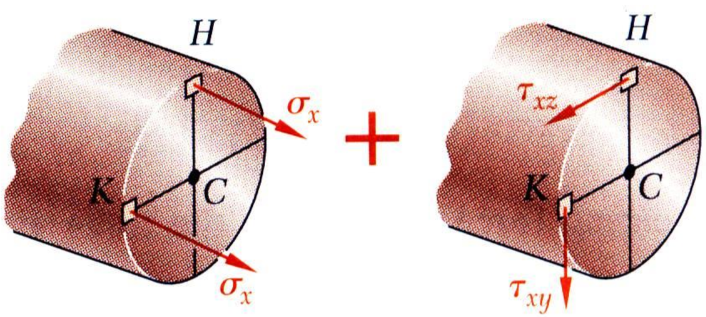
- Wish to determine stresses in slender structural members subjected to arbitrary loadings.
- Pass section through points of interest. Determine force-couple system at centroid of section required to maintain equilibrium.
- System of internal forces consist of three force components and three couple vectors.
- Determine stress distribution by applying the superposition principle.

Stresses Under Combined Loadings

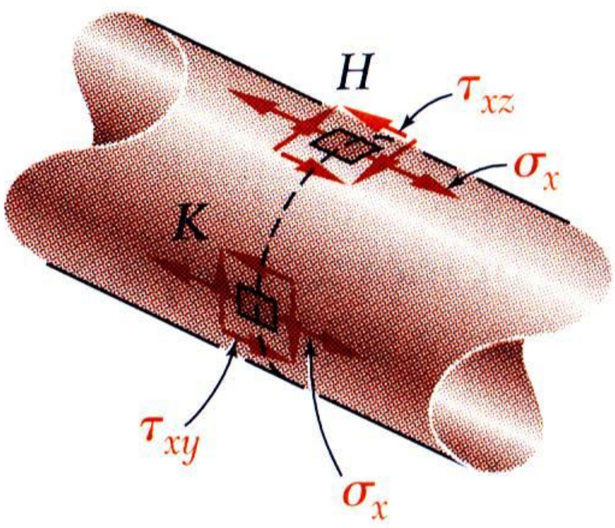


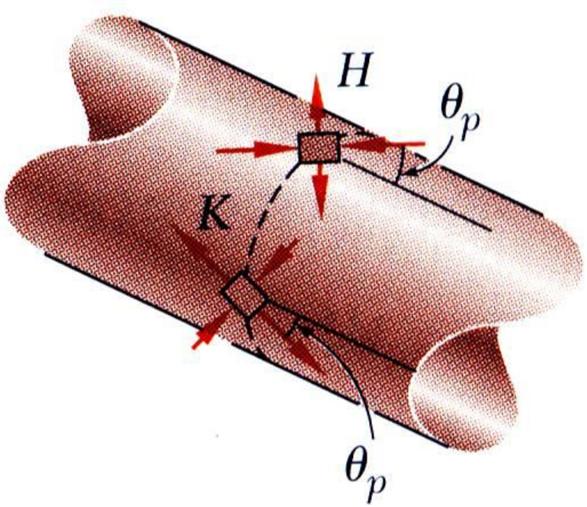
- Axial force and in-plane couple vectors contribute to normal stress distribution in the section.

- Shear force components and twisting couple contribute to shearing stress distribution in the section.



Stresses Under Combined Loadings

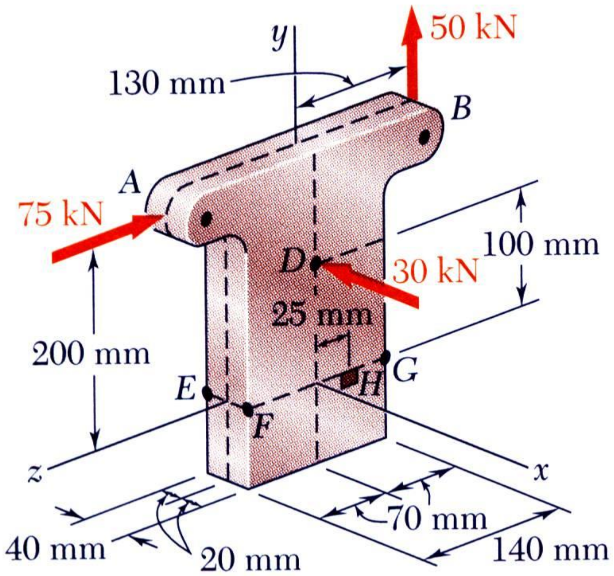




- Normal and shearing stresses are used to determine principal stresses, maximum shearing stress and orientation of principal planes.

- Analysis is valid only to extent that conditions of applicability of superposition principle and Saint-Venant's principle are met.

Sample Problem 8.5

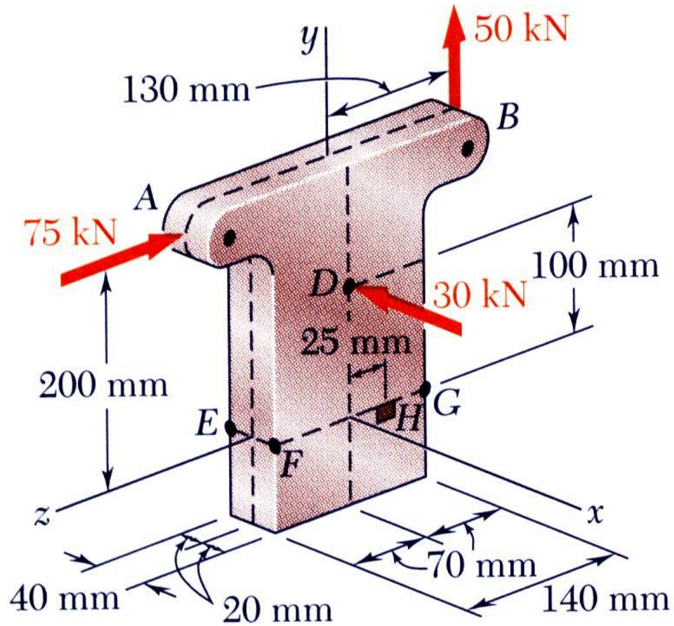


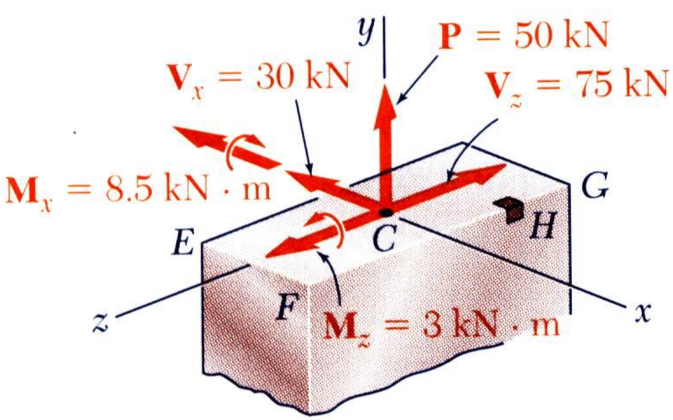
Three forces are applied to a short steel post as shown. Determine the principle stresses, principal planes and maximum shearing stress at point *H*.

SOLUTION:

- Determine internal forces in Section *EFG*.
- Evaluate normal stress at *H*.
- Evaluate shearing stress at *H*.
- Calculate principal stresses and maximum shearing stress. Determine principal planes.

Sample Problem 8.5





SOLUTION:

- Determine internal forces in Section *EFG*.

$$V_x = -30\,\text{kN} \quad P = 50\,\text{kN} \quad V_z = -75\,\text{kN}$$

$$M_x = (50\,\text{kN})(0.130\,\text{m}) - (75\,\text{kN})(0.200\,\text{m})$$

$$= -8.5\,\text{kN}\cdot\text{m}$$

$$M_y = 0 \quad M_z = (30\,\text{kN})(0.100\,\text{m}) = 3\,\text{kN}\cdot\text{m}$$

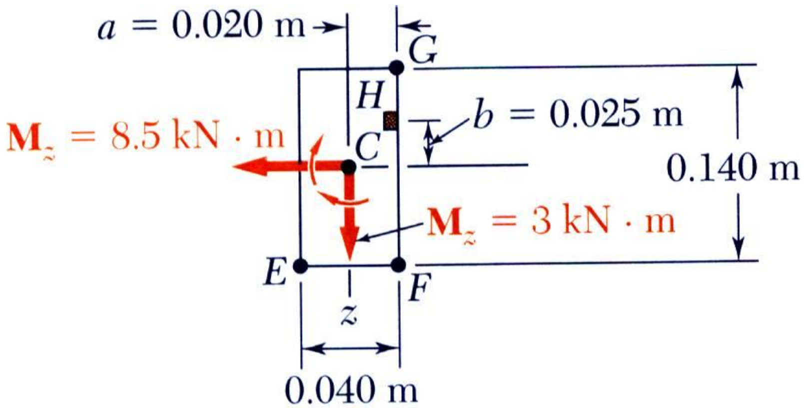
Note: Section properties,

$$A = (0.040\,\text{m})(0.140\,\text{m}) = 5.6\times10^{-3}\,\text{m}^2$$

$$I_x = \tfrac{1}{12}(0.040\,\text{m})(0.140\,\text{m})^3 = 9.15\times10^{-6}\,\text{m}^4$$

$$I_z = \tfrac{1}{12}(0.140\,\text{m})(0.040\,\text{m})^3 = 0.747\times10^{-6}\,\text{m}^4$$

Sample Problem 8.5

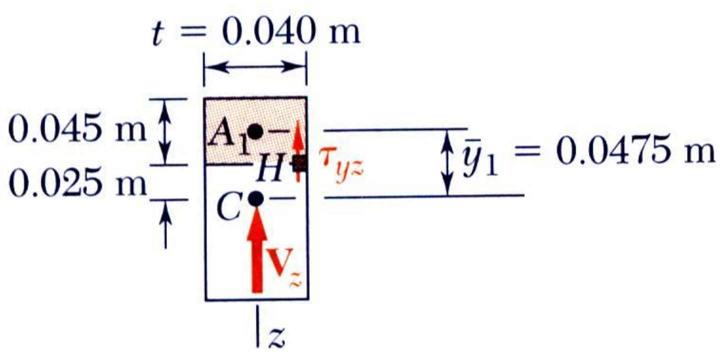


- Evaluate normal stress at *H*.

$$\sigma_y = +\frac{P}{A} + \frac{|M_z|a}{I_z} - \frac{|M_x|b}{I_x}$$

$$= \frac{50\,\text{kN}}{5.6\times 10^{-3}\,\text{m}^2} + \frac{(3\,\text{kN}\cdot\text{m})(0.020\,\text{m})}{0.747\times 10^{-6}\,\text{m}^4} - \frac{(8.5\,\text{kN}\cdot\text{m})(0.025\,\text{m})}{9.15\times 10^{-6}\,\text{m}^4}$$

$$= (8.93 + 80.3 - 23.2)\,\text{MPa} = 66.0\,\text{MPa}$$



- Evaluate shearing stress at *H*.

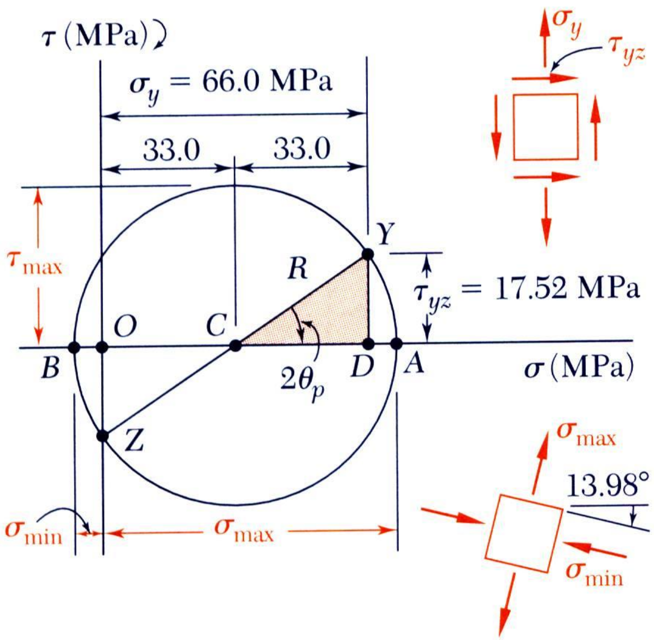
$$Q = A_1\bar{y}_1 = [(0.040\,\text{m})(0.045\,\text{m})](0.0475\,\text{m})$$

$$= 85.5\times 10^{-6}\,\text{m}^3$$

$$\tau_{yz} = \frac{V_z Q}{I_x t} = \frac{(75\,\text{kN})(85.5\times 10^{-6}\,\text{m}^3)}{(9.15\times 10^{-6}\,\text{m}^4)(0.040\,\text{m})}$$

$$= 17.52\,\text{MPa}$$

Sample Problem 8.5



- Calculate principal stresses and maximum shearing stress.
  Determine principal planes.

$$\tau_{\max} = R = \sqrt{33.0^2 + 17.52^2} = 37.4\,\text{MPa}$$

$$\sigma_{\max} = OC + R = 33.0 + 37.4 = 70.4\,\text{MPa}$$

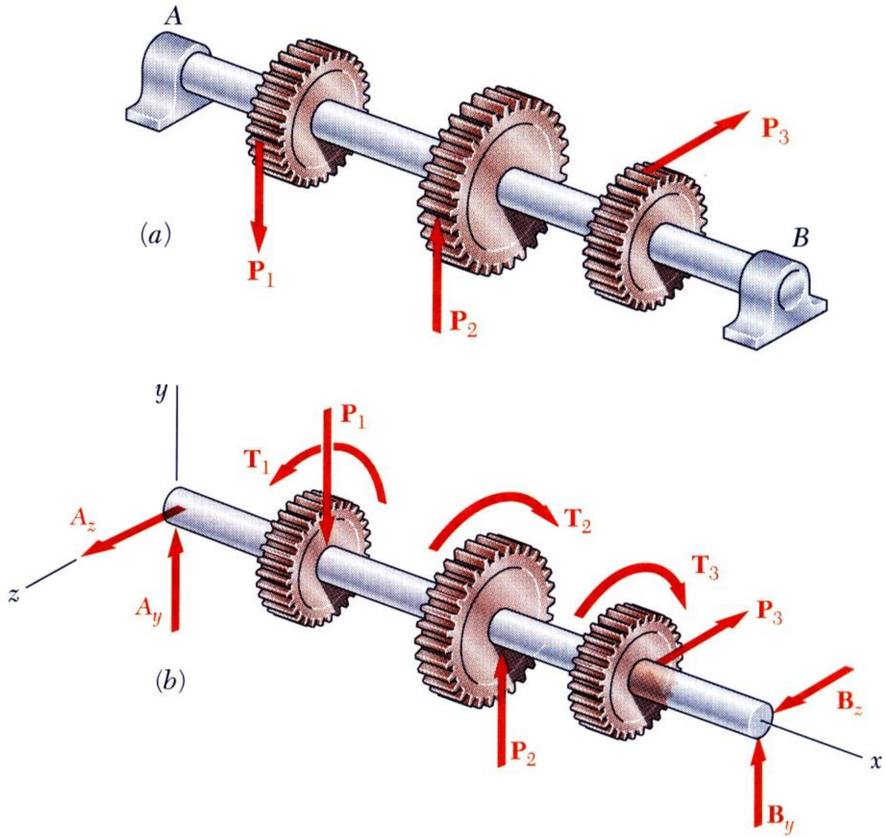
$$\sigma_{\min} = OC - R = 33.0 - 37.4 = -7.4\,\text{MPa}$$

$$\tan 2\theta_p = \frac{CY}{CD} = \frac{17.52}{33.0} \qquad 2\theta_p = 27.96^\circ$$

$$\theta_p = 13.98^\circ$$

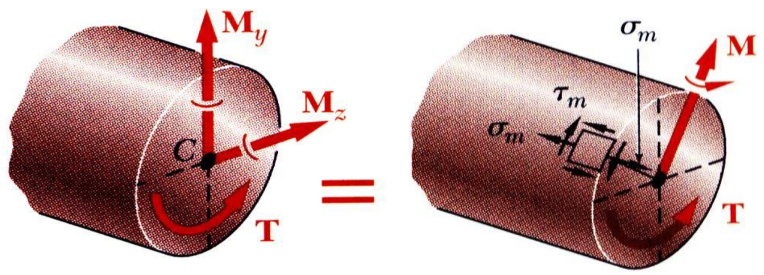
$$\boxed{\begin{aligned} \tau_{\max} &= 37.4\,\text{MPa} \\ \sigma_{\max} &= 70.4\,\text{MPa} \\ \sigma_{\min} &= -7.4\,\text{MPa} \\ \theta_p &= 13.98^\circ \end{aligned}}$$

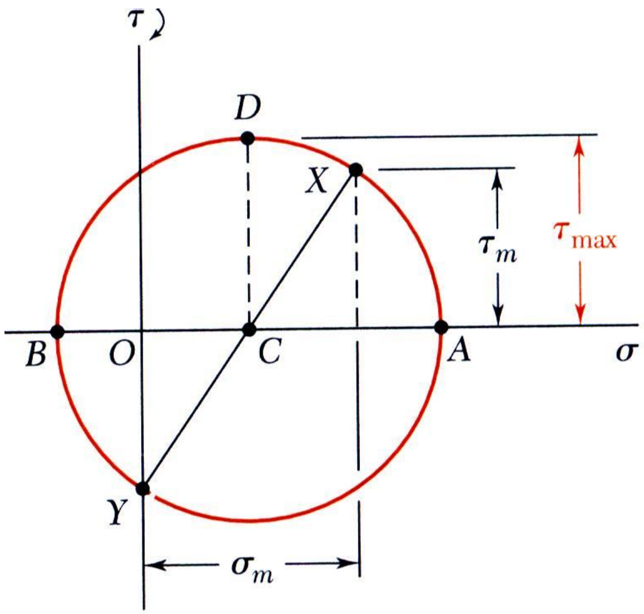
Design of a Transmission Shaft



- If power is transferred to and from the shaft by gears or sprocket wheels, the shaft is subjected to transverse loading as well as shear loading.
- Normal stresses due to transverse loads may be large and should be included in determination of maximum shearing stress.
- Shearing stresses due to transverse loads are usually small and contribution to maximum shear stress may be neglected.

Design of a Transmission Shaft





- At any section,

$$\sigma_m = \frac{Mc}{I} \quad \text{where} \quad M^2 = M_y^2 + M_z^2$$

$$\tau_m = \frac{Tc}{J}$$

- Maximum shearing stress,

$$\tau_{max} = \sqrt{\left(\frac{\sigma_m}{2}\right)^2 + (\tau_m)^2} = \sqrt{\left(\frac{Mc}{2I}\right)^2 + \left(\frac{Tc}{J}\right)^2}$$

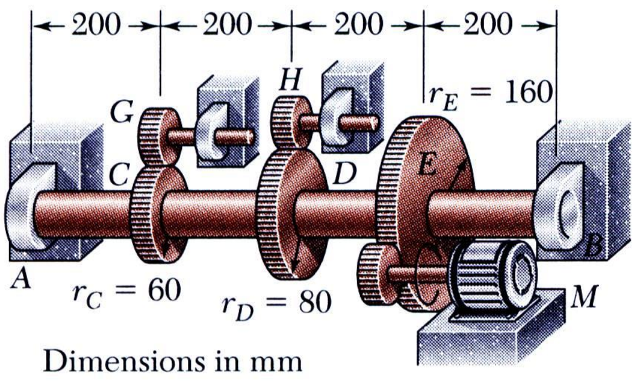
for a circular or annular cross-section, $2I = J$

$$\tau_{max} = \frac{c}{J}\sqrt{M^2 + T^2}$$

- Shaft section requirement,

$$\left(\frac{J}{c}\right)_{min} = \frac{\left(\sqrt{M^2 + T^2}\right)_{max}}{\tau_{all}}$$

Sample Problem 8.3

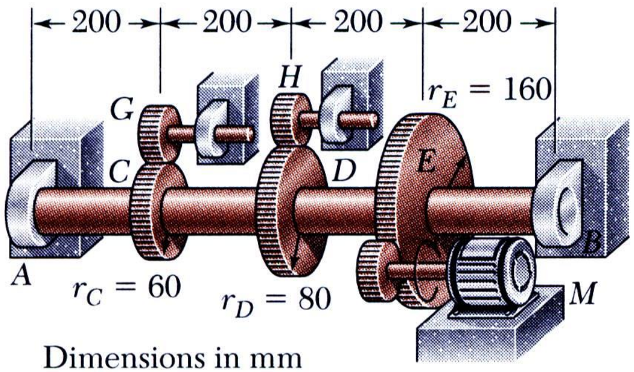


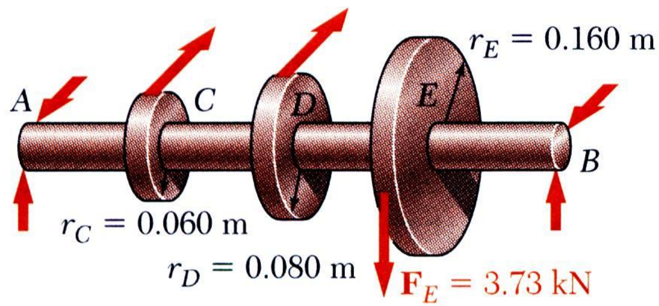
Solid shaft rotates at 480 rpm and transmits 30 kW from the motor to gears *G* and *H*; 20 kW is taken off at gear *G* and 10 kW at gear *H*. Knowing that σ_{all} = 50 MPa, determine the smallest permissible diameter for the shaft.

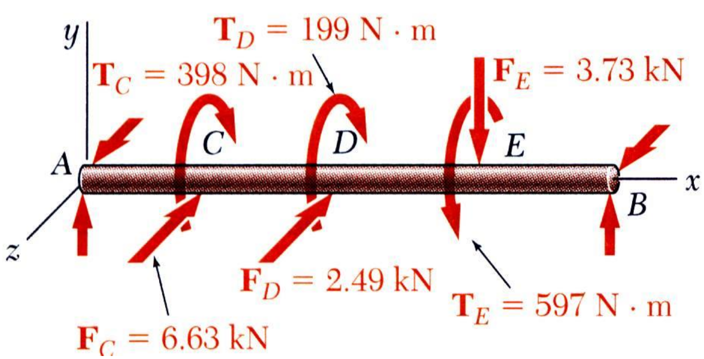
SOLUTION:

- Determine the gear torques and corresponding tangential forces.
- Find reactions at *A* and *B*.
- Identify critical shaft section from torque and bending moment diagrams.
- Calculate minimum allowable shaft diameter.

Sample Problem 8.3







SOLUTION:

- Determine the gear torques and corresponding tangential forces.

$$T_E = \frac{P}{2\pi f} = \frac{30\,\text{kW}}{2\pi(80\,\text{Hz})} = 597\,\text{N}\cdot\text{m}$$

$$F_E = \frac{T_E}{r_E} = \frac{597\,\text{N}\cdot\text{m}}{0.16\,\text{m}} = 3.73\,\text{kN}$$

$$T_C = \frac{20\,\text{kW}}{2\pi(80\,\text{Hz})} = 398\,\text{N}\cdot\text{m} \qquad F_C = 6.63\,\text{kN}$$

$$T_D = \frac{10\,\text{kW}}{2\pi(80\,\text{Hz})} = 199\,\text{N}\cdot\text{m} \qquad F_D = 2.49\,\text{kN}$$

- Find reactions at A and B.

$$A_y = 0.932\,\text{kN} \qquad A_z = 6.22\,\text{kN}$$

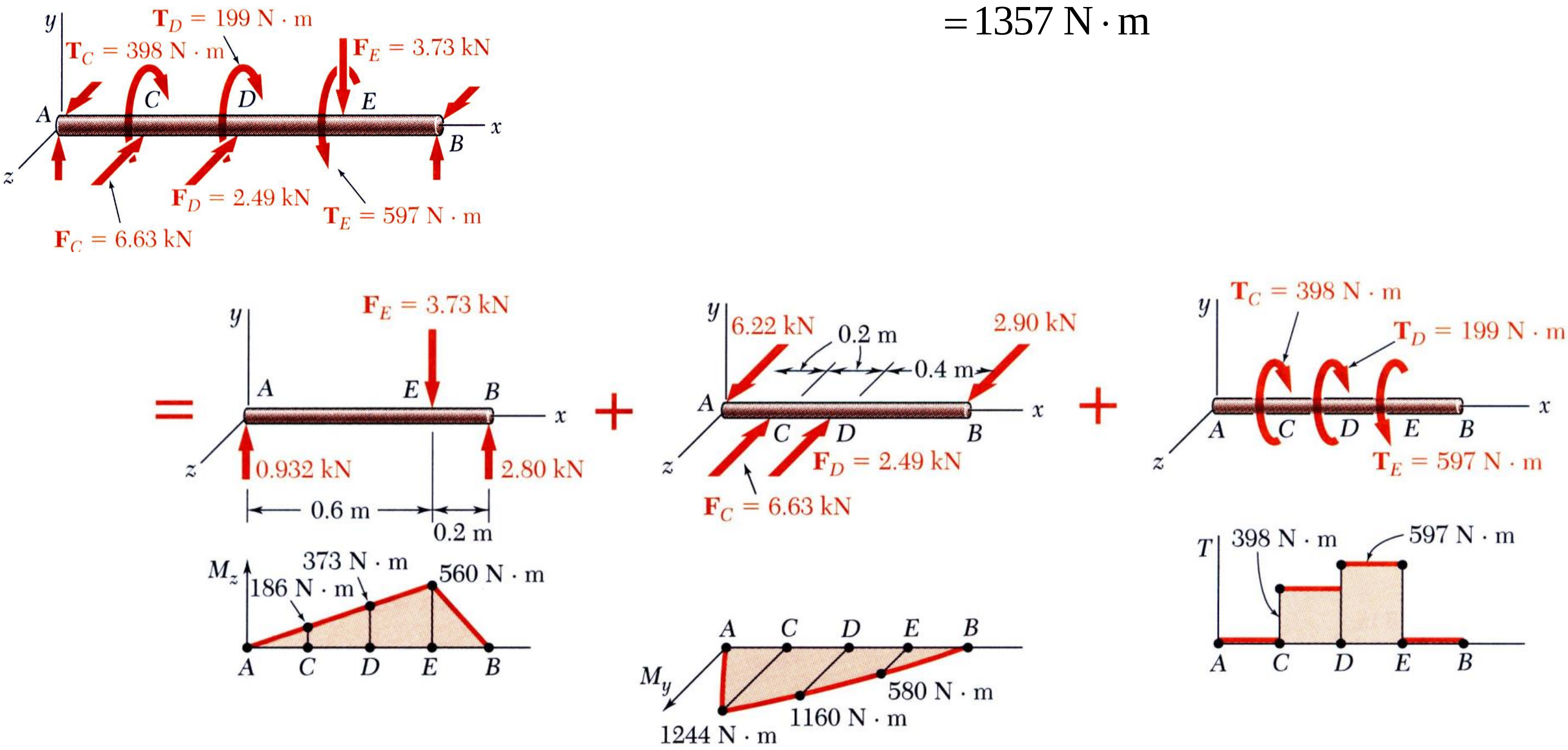
$$B_y = 2.80\,\text{kN} \qquad B_z = 2.90\,\text{kN}$$

Sample Problem 8.3

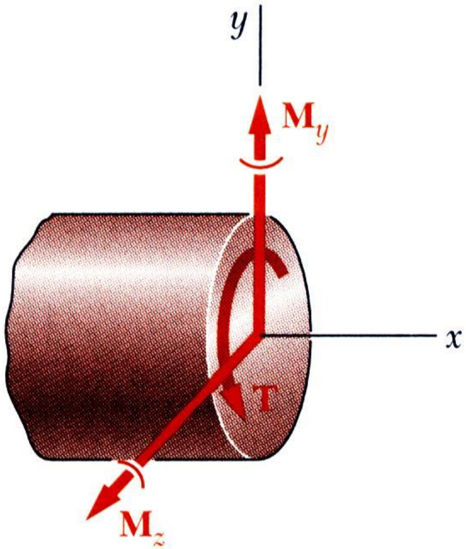
- Identify critical shaft section from torque and bending moment diagrams.

$$\left(\sqrt{M^2+T^2}\right)_{\max} = \sqrt{\left(1160^2+373^2\right)+597^2}$$

$$= 1357\ \text{N}\cdot\text{m}$$



Sample Problem 8.3



- Calculate minimum allowable shaft diameter.

$$\frac{J}{c} = \frac{\sqrt{M^2 + T^2}}{\tau_{all}} = \frac{1357\ \mathrm{N \cdot m}}{50\ \mathrm{MPa}} = 27.14 \times 10^{-6}\,\mathrm{m}^3$$

For a solid circular shaft,

$$\frac{J}{c} = \frac{\pi}{2} c^3 = 27.14 \times 10^{-6}\,\mathrm{m}^3$$

$$c = 0.02585\ \mathrm{m} = 25.85\ \mathrm{m}$$

$$\boxed{d = 2c = 51.7\ \mathrm{mm}}$$